Stochastic resonance in two coupled fractional oscillators with potential and coupling parameters subjected to quadratic asymmetric dichotomous noise

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Exact analytical expressions for the average output amplitude gains in the very long-time limit for two coupled oscillators governed by fractional-order intrinsic and external damping, under the influence of a multiplicative quadratic asymmetric dichotomous noise affecting the two potential parameters and the coupling factors, and subjected to a noise-free or noise-modulated external periodic force with same frequency, have been derived. The trustworthiness of the analytical expressions has been checked by comparing the numerical results obtained using these for some typical cases with corresponding findings based on numerical simulations. The numerical simulations have also been used to investigate the time-evolution of a representative system in the transient state by studying the probability density and displacements of the two oscillators at different times and for different values of noise intensity. Analytical expressions have been used to obtain plots for gains versus noise intensity to study the effect of (i) variation in mass parameter of one oscillator keeping that of other the same, and (ii) change in the relative values of the two coupling coefficients. Furthermore, the special case where both the potential parameters are taken to be zero, which corresponds to rectilinear motion of the two particles in the absence of fluctuations, has been examined under the influence of the second-order noise and stochastic resonance has been found to occur at lower frequencies of the applied force. This highlights the importance of nonlinear term in the noise, which makes the rectilinear motion to be oscillatory. Also, the key role played by the coupling in this has been brought out.

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1. Introduction

Literally, word noise signifies an irregular, noncontrollable and unwanted signal which hampers proper knowledge and, hence, it must be eliminated to the best possible extent. In different scientific context, it represents temporal or spatial random fluctuations which are ubiquitous in all physical as well as biological systems, and have their origin in the disturbances within the system itself and its surroundings with which it is interacting. In these cases, the effect of noise becomes conspicuous when the system is away from equilibrium or is microscopic/nanosscopic in size. In contrast with earlier notion, it has been found that stochasticity produced by noise can play constructive role in the evolution

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of a dynamical system by producing an ordered or organized state rather than a disordered one [1,2]. Some typical examples of its unexpected positive contributions are additive and multiplicative noise-induced stochastic resonance [[3–9] and references therein]; multiplicative noise generated enhanced temporal oscillations, spatial pattern formation, and noise-delayed extinction of species in population dynamics of spatially extended ecological systems [10–13]; the ratchet effect [14]; phase transitions [1,2]; stabilization by dissipation and stochastic resonant activation in metastable systems [15,16]; and nonlinear relaxation in condensed matter systems through metastable states [17,18].

One of the most investigated phenomenon displaying counterintuitive role played by the fluctuations or randomness is stochastic resonance (SR), which essentially refers to utilization of energy absorbed from the noise for amplification of the response of a system subjected to a weak periodic driving signal. Beginning from the usage of the concept in 1981 as a possible mechanism to account for the nearly periodic recurrence of ice ages on the earth and the first laboratory demonstration in Schmitt triggers electronic circuit in 1983, its development gained momentum after 1995 and now it has become an interdisciplinary field of research covering a wide spectrum of applications in classical as well as quantum physics, chemistry, materials science, climatology, ecology, neuroscience, medical science, lasers, various branches of engineering and financial management [[1,3–9,19–21], and references there].

Initially, it was believed that the SR can be observed only in nonlinear systems under the influence of periodic force and an additive noise [[3–5], and references there]. However, later works established that even linear systems can exhibit SR in the presence of periodic signal together with multiplicative coloured noise [22,23], and one of the most commonly studied example of its manifestation is the theoretician’s favourite harmonic oscillator subjected to an external periodic force and coloured-noise, where the oscillator is taken to be either underdamped or overdamped and fluctuations are assumed to affect one or more of the oscillator parameters — mass, damping, and internal frequency [24–28]. Very recently, it has been numerically shown that SR can also occur in periodic potential linear systems under the influence of periodic driving force and additive white random fluctuations (see [29], and the references therein). Besides, it has been argued that this phenomenon can manifest itself even in the presence of purely entropic barriers in higher dimensional set-ups [30] and that it can be treated as a merely geometric effect [31].

Furthermore, guided by the fact that numerous real oscillatory systems, such as those having environment like viscoelastic media, dense polymer solutions, colloidal systems, magneto-rheological fluids, and amorphous semiconductors are also influenced by their history of movement and non-local distributed features; these problems involving memory effects have been analysed by replacing the velocity and acceleration terms in the Langevin equation for the oscillator by the corresponding non-integer time derivatives [32–34]. These are, respectively, referred to as the fractional-order external and intrinsic damping terms because these are associated with the energy dissipation arising from interaction of the oscillator with the surrounding medium and consumption of energy leading to attenuation due to damping produced by the oscillator itself [35]. Also, both these types of damping are governed by power-law. As such, recently, numerous works have been reported in the literature where an oscillator with only fractional-order external damping term and with both the external as well as the intrinsic damping have been investigated under the influence of a multiplicative coloured noise. In fact, it has been found that the models involving fractional derivatives are more realistic than those based on the conventional integer-order derivatives, particularly for describing the physical phenomena in complex heterogeneous systems and complex disordered materials [36,37].

It is important to note that major emphasis of the research pursuits in this direction has been at studying single oscillator [24–28,37–43] or large but finite number of coupled oscillators [44–46] with terms having integer as well as fractional derivatives. The interest in the coupled oscillators has been motivated by the search for the effect of coupling on enhancement of SR and its practical applications in biological systems. Surprisingly, the case of two-coupled oscillators subjected to multiplicative coloured noise has not been much explored. A publication in this direction has appeared very recently, where SR of two identical oscillators with fluctuating mass under the influence of symmetric dichotomous noise and having only fractional external damping term has been studied [47]. The choice of two oscillators with the same parameter values enabled the authors to consider the motion of only one oscillator as the average behaviour of the two particles is completely synchronous. However, it may be pointed out that some workers have investigated two-coupled quantum oscillators from a different perspective and have highlighted the importance of these systems [48,49].

In view of the above stated facts, we have undertaken the investigation of SR in two coupled oscillators with intrinsic as well as external power-law damping, employing the fractional generalized Langevin equation, and fluctuating potential parameters and linear coupling parameters under the influence of the same and realistic multiplicative quadratic asymmetric dichotomous noise and an applied periodic force which is either noise-free or noise-modulated. The exact solution for the equations in the infinitely long-time limit, obtained through the Laplace transform method, has been used to bring out the effects of variation in the mass, the potential, and the coupling parameters on the stationary state output amplitude gains with emphasis on their comparison. However, prior to executing this task, we have ascertained the reliability of the findings based on the analytical expressions by comparing these for some typical cases with corresponding results obtained from numerical simulations. In addition, the numerical simulations have also been used to study the time-evolution of a representative system in the transient state by investigating the probability density over the first time-period of the driving force and displacements of the two oscillators over the two time-periods, at different values of time as well as noise intensity.
2. System model

We consider a system consisting of two coupled different oscillators with fractional-order intrinsic as well as external damping terms under the influence of multiplicative quadratic asymmetric dichotomous noise (ADN), affecting both the potential as well as the coupling parameters, in the presence of two external periodic forces having the same frequency, one of which is also modulated by a noise. The fractional generalized Langevin equations describing the system read

\[ m_1D^n x_1(t) + γ_1 D^n x_1(t) + [k_1 + a_1 μ(t) + a_2 μ^2(t)]x_1(t) = [C_1 + b_1 μ(t) + b_2 μ^2(t)]x_1(t) + [A + B_1 ξ(t)] cos(Ωt), \]

(1)

and

\[ m_2 D^n x_2(t) + γ_2 D^n x_2(t) + [k_2 + d_1 μ(t) + d_2 μ^2(t)]x_2(t) = [C_2 + e_1 μ(t) + e_2 μ^2(t)]x_2(t) + [A + B_2 ξ(t)] cos(Ωt). \]

(2)

Here, \( x_j(t) \), \( m_j \), \( γ_j \), \( k_j \), and \( C_j \) are the instantaneous displacements, mass parameter, friction parameter, potential parameter and coupling coefficient, respectively, of the \( j \)th \((j = 1, 2)\) oscillator. It may, however, be mentioned that \( m_j \) and \( γ_j \), respectively, have dimension \( MT^α–2 \) and \( MT^{1–β} \). Also, \( D^α \) with \( α = β \) is the Caputo time fractional derivative operator of order \( ρ \) [50], viz.,

\[ D^α x(t) = \left[ \frac{1}{Γ(n–ρ)} \right] \int_0^t (t − t')^{n–1–ρ} \frac{d^n}{dt^n} x(t')dt', \]

(3)

where \( (n – 1) < ρ ≤ n, n ∈ N \) and \( 1 < α ≤ 2, 0 < β ≤ 1 \). Thus, the first two terms are the expressions for the power-law decay kernel based intrinsic and external damping, respectively. Also, \( α = 2 \) gives ordinary acceleration while \( β = 1 \) corresponds to usual velocity, and for \( α = 2 \), and \( β = 1 \) Eqs. (1) and (2) become classical Langevin equations for a pair of coupled harmonic oscillators in the presence of the fluctuations considered here. \( a_1 μ(t) + a_2 μ^2(t) \) and three other similar expressions in the two equations represent relevant quadratic multiplicative asymmetric dichotomous noise \( μ(t) \) with the first term as the linear part and the second term as the nonlinear part so that the noise becomes linear if the coefficient in the latter is zero. The noise has been taken to be quadratic in nature because this provides a better description of the dynamical properties of a system [51]. Also, the coefficients of the linear as well as nonlinear terms in the noise will be assumed to be non-negative in all the expressions. Furthermore, though the noise acting on the potential and the coupling parameters is the same, its contributions to different parameters have been taken to be different. \([A + B_1 ξ(t)] cos(Ωt)\) denotes the sum of two applied periodic forces having same frequency \( Ω \), with the second, viz. \( B_2 ξ(t) cos(Ωt) \), being under the influence of ADN, \( ξ(t) \) different from the one causing fluctuations in the potential and the coupling parameters of the two oscillators. Here, we have used sum of the two forces in the derivation, but only one is taken to be operative at a time. The amplitude \( B = 0 \) leads to noise-free external force, while amplitude \( A = 0 \) corresponds to only noise-modulated external signal. We have ignored the additive internal noise. It may be added that \( C_1 = C_2 = b_1 = b_2 = e_1 = e_2 = 0 \) lead to the case of two uncoupled fractional oscillators.

The ADN, \( μ(t) \) consists of jumps between two values: \( −Δ_μ \) and \( Γ_μ \) (both \( Δ_μ \) and \( Γ_μ > 0 \)) with transition rate \( γ_μ \) from \( −Δ_μ \) to \( Γ_μ \) and \( γ'_μ \) for the reverse jump from \( Γ_μ \) to \( −Δ_μ \). Thus, different parameters characterizing this ADN are [43]

Noise intensity \( ≡ A_μ = Γ_μ Δ_μ \),

Noise correlation rate \( ≡ ν_μ = γ_μ + γ'_μ \),

(4)

and

Noise asymmetry \( ≡ δ_μ = Γ_μ – Δ_μ \).

It may be mentioned that

\[ μ^2(t) = A_μ + δ_μ μ(t) \]

(5)

and that \( δ_μ = 0 \) makes the noise symmetric DN. Also, in the stationary state, \( μ(t) \) is a zero mean exponentially correlated noise, implying that

\[ ⟨μ(t)⟩ = 0, \]

(6a)

and

\[ ⟨μ(t)μ(t')⟩ = A_μ e^{−ν_μ |t−t'|}. \]

(6b)

Replacing the subscript \( μ \) by \( ξ \) in the preceding part, we get the corresponding features of the ADN \( ξ(t) \). Also, the two noises are taken such that

\[ ⟨μ(t)ξ(t)⟩ = A_μ ξ, \]

(7)

where the cross correlation, \( A_μ ξ \), is determined by the intensities and the coupling of the two ADN.
Also, from Eq (5), we have
\[ \langle \mu^2(t)x_j(t) \rangle = \Lambda_{\mu}(x_j(t)) + \delta_{\mu_j}(\mu(t)x_j(t)), \]
(8)
and
\[ \langle \mu^3(t)x_j(t) \rangle = \Lambda_{\mu}\delta_{\mu_j}(x_j(t)) + (\Lambda_{\mu} + \delta^2_{\mu}(\mu(t)x_j(t)). \]
(9)
Furthermore,
\[ (\mu(t)D^\nu x_j(t)) = e^{-\nu t}D^\nu [(\mu(t)x_j(t))e^{\nu t}], \]
(10)
which is generalization of the Shapiro–Loginov formulae of differentiation [52], corresponding to the fractional derivative of order $\rho$ ($= \alpha$ as well as $\beta$) [39].

3. Exact solution and output amplitude gains

In order to determine the first moments $\langle x_1(t) \rangle$ and $\langle x_2(t) \rangle$, we find averages over all the realizations of the trajectories for the simultaneous stochastic equations (1) and (2). Using (6) and (8), and making the following substitutions for later convenience,
\[ \langle x_1(t) \rangle = y_1(t), \langle x_2(t) \rangle = y_2(t), \langle \mu(t)x_1(t) \rangle = y_3(t), \text{and} \langle \mu(t)x_2(t) \rangle = y_4(t), \]
(11)
we get
\[ \left[ m_1 D^\nu + y_1 D^\rho + [k_1 + a_2 \Lambda_{\mu}])y_1(t) + (a_1 + a_2 \delta_{\mu_j})y_3(t) \right. \]
\[ = (C_1 + b_2 \Lambda_{\mu})y_2(t) + (b_1 + b_2 \delta_{\mu_j})y_4(t) + A \cos(\Omega t), \]
(12)
and
\[ \left[ m_2 D^\nu + y_2 D^\rho + (k_2 + d_2 \Lambda_{\mu})y_2(t) + (d_1 + d_2 \delta_{\mu_j})y_4(t) \right. \]
\[ = (C_2 + b_2 \Lambda_{\mu})y_1(t) + (b_1 + b_2 \delta_{\mu_j})y_3(t) + A \cos(\Omega t), \]
(13)
To take care of the additional correlators $y_j(t) = \langle \mu(t)x_j(t) \rangle$, ($l = 3, 4$ corresponding to $j = 1, 2$, respectively), we multiply both sides of (1) and (2) with $\mu(t)$, average the resulting equations and use Eqs. (7)–(10). This gives us
\[ (a_1 + a_2 \delta_{\mu_j})\Lambda_{\mu_j}y_1(t) + e^{-\nu t} [m_1 D^\nu + y_1 D^\rho]y_3(t) + [k_1 + a_1 \delta_{\mu_j} + a_2(\Lambda_{\mu} + \delta^2_{\mu})]y_3(t) \]
\[ = (b_1 + b_2 \delta_{\mu_j})\Lambda_{\mu_j} y_2(t) + [C_1 + b_1 \delta_{\mu_j} + b_2(\Lambda_{\mu} + \delta^2_{\mu})]y_4(t) + A_{\mu} B \cos(\Omega t), \]
(14)
and
\[ (d_1 + d_2 \delta_{\mu_j})\Lambda_{\mu_j} y_2(t) + e^{-\nu t} [m_2 D^\nu + y_2 D^\rho]y_4(t) + [k_2 + d_1 \delta_{\mu_j} + d_2(\Lambda_{\mu} + \delta^2_{\mu})]y_4(t) \]
\[ = (b_1 + b_2 \delta_{\mu_j})\Lambda_{\mu_j} y_1(t) + [C_2 + b_1 \delta_{\mu_j} + b_2(\Lambda_{\mu} + \delta^2_{\mu})]y_3(t) + A_{\mu} B \cos(\Omega t). \]
(15)
Following the usual practice, we solve Eqs. (12)–(15) using Laplace transform method and denote the Laplace transform of $y_3(t)$ by $Y_3(s), (n = 1–4)$. Since we are ultimately interested in the solutions in the very long-time limit ($t \to \infty$), the memory effects associated with the initial conditions $y_3(0)$ and $y_4(0), (n = 1–4)$, which appear during the process of determining the Laplace transforms, can be ignored. Accordingly, we obtain
\[ T_{11}Y_1(s) + T_{12}Y_2(s) + T_{13}Y_3(s) + T_{14}Y_4(s) = \Lambda_B = \frac{s}{s^2 + \Omega^2}, \]
(16)
\[ T_{21}Y_1(s) + T_{22}Y_2(s) + T_{23}Y_3(s) + T_{24}Y_4(s) = \Lambda_B = \frac{s}{s^2 + \Omega^2}, \]
(17)
\[ T_{31}Y_1(s) + T_{32}Y_2(s) + T_{33}Y_3(s) + T_{34}Y_4(s) = \Lambda_B = \frac{s}{s^2 + \Omega^2}, \]
(18)
and
\[ T_{41}Y_1(s) + T_{42}Y_2(s) + T_{43}Y_3(s) + T_{44}Y_4(s) = \Lambda_B = \frac{s}{s^2 + \Omega^2}. \]
(19)
The coefficients $T_{ij}$ ($i, n = 1–4$) have been listed in Appendix A.

Using Cramer’s rule to solve the above set of linear algebraic equations in $Y_n(s)$, we get for $n = 1, 2,$
\[ Y_1(s) = \left[ H_{13}(s)A + H_{14}(s)A_{\mu}B \right] \frac{s}{s^2 + \Omega^2}, \]
(20)
and
\[ Y_2(s) = \left[ H_{23}(s)A + H_{24}(s)A_{\mu}B \right] \frac{s}{s^2 + \Omega^2}. \]
(21)
The coefficients $H_{na}(s)$ and $H_{nb}(s)$ introduced above are defined in Appendix B. The inverse Laplace transform of $Y_{n}(s)$, $n = 1, 2$, yields expression for $y_{n}(t) = \langle x_{n}(t) \rangle$ in the $t \to \infty$ limit. Thus, in the very long-time limit, i.e. the stationary state, when only the applied periodic force of frequency $\Omega$ is operative, the average displacement of an oscillator is given by

$$
y_{n}(t)|_{t \to \infty} = \langle x_{n}(t) \rangle|_{t \to \infty} = A \int_{0}^{\infty} h_{na}(t - t')\cos(\Omega t')dt' + A_{\mu A}B \int_{0}^{\infty} h_{nb}(t - t')\cos(\Omega t')dt'.
$$  \tag{22}

Note that $h_{na}(t - t')$ and $h_{nb}(t - t')$ account for the retarded response (in the form of displacement) of the corresponding oscillator (under the influence of ADN) to the applied two forces. Accordingly, $h_{na}(t)$ and $h_{nb}(t)$, which are inverse Laplace transforms of $H_{na}(s)$ and $H_{nb}(s)$, respectively, are called relaxation functions at time $t$. In the light of linear response theory [38], we write

$$
\langle x_{n}(t) \rangle_{st} = A_{n} \cos(\Omega t + \theta_{na}) + A_{\mu B}B_{n} \cos(\Omega t + \theta_{nb}) \quad (n = 1, 2).
$$  \tag{23}

Here,

$$
A_{n} = A|H_{na}(s = -i\Omega)| = A \left[ \frac{N_{na1}^{2} + N_{na2}^{2}}{D_{1}^{2} + D_{2}^{2}} \right]^{\frac{1}{2}},
$$  \tag{24a}

$$
A_{\mu B}B_{n} = A_{\mu B}|H_{nb}(s = -i\Omega)| = A_{\mu B}B \left[ \frac{N_{nb1}^{2} + N_{nb2}^{2}}{D_{1}^{2} + D_{2}^{2}} \right]^{\frac{1}{2}},
$$  \tag{24b}

and

$$
\theta_{na} = \arctan \left[ \frac{-N_{na1}D_{2} - N_{na2}D_{1}}{N_{na1}D_{1} + N_{na2}D_{2}} \right], \quad \text{and}
$$

$$
\theta_{nb} = \arctan \left[ \frac{-N_{nb1}D_{2} - N_{nb2}D_{1}}{N_{nb1}D_{1} + N_{nb2}D_{2}} \right].
$$  \tag{25a}

Eqs. (24) and (25) define amplitudes and phase shifts of the output $\langle x_{n}(t) \rangle_{st}$ of the $n$th oscillator produced by the effect of applied two forces, $A_{n} \cos(\Omega t)$ and $B_{n} \cos(\Omega t)$, respectively. Also, $N_{na1}$ and $N_{na2}$ are the real and imaginary parts of the numerator of $H_{na}(s = -i\Omega)$, while $D_{1}$ and $D_{2}$ are similar quantities pertaining to its denominator, which is $D(s = -i\Omega)$. Similar statement holds good for $H_{nb}(s = -i\Omega)$, by replacing $A$ by $B$. Therefore, the relevant stationary state output amplitude gains are given by

$$
GA_{n} = \frac{A_{n}}{A} \quad (B = 0); \quad \text{and} \quad GB_{n} = \frac{A_{\mu B}B_{n}}{B} \quad (A = 0).
$$  \tag{26}

It must be emphasized that the average displacement $\langle x_{n}(t) \rangle_{st}$, Eq. (23), has only the first term for the noise-free external force and only the second term when the noise-modulated force is acting.

It is pertinent to note that the fractional differential equations governing the average displacement of the two oscillators, viz. Eqs. (12) and (13), contain terms $(k_{1} + a_{2} \Lambda_{\mu})y_{1}(t)$ and $(k_{2} + d_{2} \Lambda_{\mu})y_{2}(t)$, respectively, which imply that the nonlinear term in the noise enhances the values of the potential parameters $k_{1}$ and $k_{2}$ (for $a_{2}$ and $d_{2} > 0$). Moreover, these noise-modulated effective potential parameters do not vanish even when the potential parameters $k_{1} = k_{2} = 0$, which correspond to the situation of coupling between two linearly moving particles under the influence of ADN. Of course, coupling is not necessary and the presence of the second-order term in the multiplicative noise influencing $x(t)$ is enough to render the rectilinear motion of a particle to be oscillatory with frequency proportional to the square root of the product of the coefficient of nonlinear term in noise and the noise intensity. Obviously, it is interesting to explore the possibility of observing SR in the system having $k_{1} = k_{2} = 0$ and to investigate the contribution of coupling to this, which has been found to be crucial (see 5.3). Furthermore, though the contribution of linear term in noise does not appear in the coefficients of $y_{1}(t)$ and $y_{2}(t)$ in (12) and (13), its coefficients do occur in the expressions for $GA_{n}$ and $GB_{n}$ because of their presence in the relevant terms in (14) and (15) and hence these coefficients are expected to affect the gains.

However, before proceeding to the next section, it may be pointed out that the solution (22) will be stable, and hence causal [38,53] if the noise intensity $\Lambda_{\mu}$ is such that

$$
0 < \Lambda_{\mu} < (\Lambda_{\mu})_{ct},
$$  \tag{27}

where $(\Lambda_{\mu})_{ct}$, the critical value of noise intensity, is the smallest positive root of the $D(s = 0) = 0$, with $D(s)$ defined in Appendix B. This being a complicated quartic equation, we have determined $(\Lambda_{\mu})_{ct}$ graphically by plotting $D(s = 0)$ as a function of $\Lambda_{\mu}$. We have used the result so obtained to ensure that the inequality (27) is satisfied for every case used to study dependence of the gains $GA_{n}$ and $GB_{n}$ on different parameters.
4. Numerical simulations

Before proceeding further with the detailed study of the numerical results obtained from the analytical expressions found for different output gains in the preceding section, viz. Eq. (26) and the related ones, we determine numerical solution of Eqs. (1) and (2), and use these, together with Monte Carlo experiment, to study two aspects of the system.

Firstly, we evaluate peak magnitudes of gains for some typical combination of different parameter values for sufficiently high value of time, t, and compare these with the corresponding numerical results found from Eq. (26), to assess the correctness of the results and discussion to be presented in Section 5. Secondly, we explore the effect of noise intensity on the time-evolution of the two oscillators in the transient state soon after starting from rest at the origin at t = 0.

To begin with, we rewrite Eqs. (1) and (2) as

\[ m_1 \ddot{x}_1(t) + \gamma_1 \dot{x}_1(t) = F_1(x_1(t), x_2(t), t) \]  

and

\[ m_2 \ddot{x}_2(t) + \gamma_2 \dot{x}_2(t) = F_2(x_2(t), x_1(t), t) \]  

where

\[ F_1(x_1, x_2, t) = -[k_1 + \{a_1 + a_2 \delta \mu\} \mu(t) + a_2 \Lambda \mu] x_1(t) + [C_1 + \{b_1 + b_2 \delta \mu\} \mu(t) + b_2 \Lambda \mu] x_2(t) + [A + B \xi(t)] \cos(\Omega t), \]  

and

\[ F_2(x_2, x_1, t) = -[k_2 + \{d_1 + d_2 \delta \mu\} \mu(t) + d_2 \Lambda \mu] x_2(t) + [C_2 + \{e_1 + e_2 \delta \mu\} \mu(t) + b_2 \Lambda \mu] x_1(t) + [A + B \xi(t)] \cos(\Omega t), \]  

Here, we have used Eq. (5) to get the explicit expressions (30) and (31).

Now, we extend the finite-difference method based time-discretization scheme for the Caputo fractional derivative formulated by Lin and Xu [54] for 0 < \beta ≤ 1 to the case 1 < \alpha ≤ 2. For this purpose, we introduce discrete time \( t_i = l \Delta t \), where \( \Delta t \) is the time-step size. Thus, the discretized version of Eqs. (28) and (29) reads

\[ m_n \ddot{x}_n(t_{i+1}) + \gamma_n \dot{x}_n(t_{i+1}) = F_n(x_n(t_{i+1}), x_l(t_{i+1}), t_{i+1}), \]  

where \((n,j) = (1, 2) \) and \( j \neq n \). Also

\[ m_n \ddot{x}_n(t_{i+1}) = \left[ \frac{m_n}{\Gamma(2 - \alpha)} \right] \sum_{l=0}^{t_{i+1}} x_n(t_{l+1}) - 2x_n(t_l) + x_n(t_{l-1}) \frac{1}{(\Delta t)^2} \int_{t_l}^{t_{i+1}} (t_{l+1} - t')^{1-\alpha} dt', \]  

and

\[ \gamma_n \dot{x}_n(t_{i+1}) = \left[ \frac{\gamma_n}{\Gamma(1 - \beta)} \right] \sum_{l=0}^{t_{i+1}} x_n(t_{l+1}) - x_n(t_l) \frac{1}{\Delta t} \int_{t_l}^{t_{i+1}} (t_{l+1} - t')^{-\beta} dt', \]  

While writing these expressions, we have ignored the local truncation error which can be rendered negligible by choosing \( \Delta t \) to be quite small. Furthermore, to obtain a recurrence formula, following the usual practice, we replace \( F_n(x_n(t_{i+1}), x_l(t_{i+1}), t_{i+1}) \) by \( F_n(x_n(t_l), x_l(t_l), t_l) \) in Eq. (32).

Evaluating the two definite integrals in Eqs. (33) and (34), rearranging different terms in these, and substituting the expressions so obtained into the form of Eq. (32), we finally get general expressions for the numerical solution of Eqs. (1) and (2). We have then used the algorithm put forward by Barik et al. [55] to generate the ADN needed in Eqs. (31) and (32), and have then proceeded with determining the value of the gains or displacements for typical combinations of the parameters.

The first step in executing the numerical simulations is the judicious choice of step size \( \Delta t \). For this purpose, we have considered the problem in the absence of noise for \( m_1 = 1.0, m_2 = 0.25, \gamma_1 = \gamma_2 = 0.3, k_1 = k_2 = 1.0, C_1 = 0.2, C_2 = 0.5, \alpha = 1.6, \beta = 0.6, \Omega = 0.5 \), and all the parameters characterizing the noise as zero. The analytic expression, Eq. (24), gave the gain values \( G_{A_{1, num}} = 1.4022, G_{A_{2, num}} = 1.5441 \). The numerical solution for \( T = 60 \) and \( \Delta t = 0.01 \) led to \( G_{A_{1, num}} = 1.4034 \) (difference in \% \( \epsilon = 0.086 \), \( G_{A_{2, num}} = 1.5450 \) (\( \epsilon = 0.065 \)), and \( G_{A_{1, num}} = 1.4030 \) (\( \epsilon = 0.057 \), \( G_{A_{2, num}} = 1.5446 \) (\( \epsilon = 0.032 \)). For \( \Delta t = 0.0075 \), respectively; the two \( \epsilon \) became nearly half of the corresponding values when \( \Delta t \) was reduced from 0.0075 to 0.005. However, for \( T = 40 \) and \( \Delta t = 0.01 \), the relevant gains were \( G_{A_{1, num}} = 1.4040 \) (\( \epsilon = 0.128 \), \( G_{A_{2, num}} = 1.5459 \) (\( \epsilon = 0.117 \)), and for \( \Delta t = 0.005 \), the value of \( \epsilon \) was nearly 0.11 for both the gain. In order to get reasonably accurate results of numerical simulations without making the computation time to be exorbitantly large, all the numerical solutions for the first part have been obtained using \( T = 60 \) and \( \Delta t = 0.0075 \). Furthermore, while incorporating the effects of noise, the Monte Carlo experiment has been performed with 11000 realizations of the ADN for each case. To accomplish this goal, we have first determined the \( \Lambda_\mu \) value for which peak is observed for either of the two \( G_{A_n} vs. \Lambda_\mu \) plots (or \( G_{B_n} vs. \Lambda_\mu \) graphs for the noise-modulated applied force)
from the analytical expressions and used this to find both the corresponding numerical values. The numerical results so obtained have been compared with the relevant analytical formulae-based values of the gains for some typical different combinations of parameter values, in Table 1. A perusal of the $\epsilon$ values for different cases, in this table, brings out the quality of agreement between the analytical and numerical values and, hence, the reliability of the numerical results found from the analytical expressions in the next section.

As mentioned above, we next proceed to examine the behaviour of the system in the transient state, i.e., for $t$ much smaller than 60 used above. We have obtained numerical solution for Eqs. (1) and (2) by assuming that both the coupled fractional oscillators start from rest at the origin at $t = 0$ and are subjected to external force $A\cos(\Omega t)$. Because of relatively low values of $t$, the computations for this part have been carried out with $\Delta t = 0.005$ and 15000 realizations of the asymmetric dichotomous noise for the oscillator and noise parameter values: $m_1 = 1.0$, $m_2 = 0.25$, $\gamma_1 = \gamma_2 = 0.5$, $k_1 = k_2 = 1.0$, $C_1 = 0.2$, $C_2 = 0.5$, $a_1 = d_1 = 1.0$, $a_2 = d_2 = 0.2$, $b_1 = e_1 = 0.2$, $b_2 = e_2 = 0.05$, $\alpha = 1.6$, $\beta = 0.3$, $\nu_0 = 0.03$, $\delta_0 = 0.0$, $\Omega = 0.5$. It is found that the effect of noise becomes observable for $t > 1$ as for $t \leq 1$ both the displacements for $A_\mu \neq 0$ are almost the same as for $A_\mu = 0$. Keeping in mind this fact, we have determined probability density $P(x(t), t)$ for the displacement of both the oscillators for $A_\mu = 0.25$, 0.5 and 1.0 at times $t = 3.0$, 8.0, 11.0 and 14.3. Note that the displacements $x_1(t)$ and $x_2(t)$ have just passed their respective initial (relatively low amplitude) peaks at $t = 3.0$, are fairly vicinal to their negative maximum displacements at $t = 8.0$, are reasonably close to zero for $t = 11.0$, and near their maximum positive value for $t = 14.3$ during their first time-period under the driving force having $\Omega = 0.5$. The histograms, representing the probability density, have been obtained using 30 bins for good resolution of the plots, and are depicted in Fig. 1. It may be pointed out that the scales appearing in different plots, particularly along the $x$-axes, are different as per distribution over the number of bins chosen. These histograms show that for a specific value of noise intensity, the probability of finding an oscillator away from its maximum probability position is the least for $t = 3.0$ and maximum for $t = 14.3$, and this trend is more conspicuous for the oscillator with lower $m$ value. In fact, it is found that the probability of an oscillator being found away from the maximum probability position becomes higher for larger $m$ values. This is understandable because the effectiveness of dichotomous noise increases with time till the attainment of steady state. Next, for a chosen $t$ value, the $x_m(t)$ for which probability density is maximum is different for different $A_\mu$ because of difference in the phase angle and the spread in the probability density is marginally affected by an increase in the noise intensity. Also, it is interesting to note that for all the noise intensities considered here, the $x_m$ value for which the probability density is maximum is slightly more than the relevant mean value for $t = 3.0$ as well as 8.0, and it is other way for $t = 11.0$ as well as 14.3; this difference is at most 8%. As an example, we may mention that for $A_\mu = 1.0$, the values of $x_1(t)$ and $x_2(t)$ at $t = 3.0$ are 0.997 and 1.429, respectively, while the corresponding probability densities are maximum at 1.019 and 1.467, and for $t = 14.3$ it is found that $x_1(t) = 1.523$, $x_2(t) = 2.127$ whereas maximum probability densities are observed at 1.484 and 1.974, respectively. A systematic scrutiny of this aspect reveals that the trend changes between $t = 8.0$ and 9.0. This behaviour seems to be associated with the fact that the two oscillators pass through transient state up to $t = 8.0$ or so and tend towards steady state after that.

In order to understand the things better, we have found $x_1(t)$ and $x_2(t)$ for $A_\mu = 0.0$, 0.25, 0.5 and 1.0 for a reasonably large number of $t$ values up to about two time-periods by considering the oscillator and ADN parameters as listed in the preceding paragraph, and have depicted these in Fig. 2. The $t$ values for which zeros and maximum magnitudes of $x_1(t)$ and $x_2(t)$ are observed differ in these four cases because of difference in the phase angles. It is found that the phase angles obtained from these data are in complete agreement with the values determined from the analytical expressions given by Eqs. (25a) and (25b). This further strengthens the faith in correctness of the numerical results obtained from the analytical expressions. A perusal of these plots brings out the following two features. First, the magnitudes of maximum displacement, $\langle |x_m(t)|_{\text{max}} \rangle$, for both the oscillators under the influence of a specific noise intensity marginally increase

<table>
<thead>
<tr>
<th>Parameter values</th>
<th>Noise Intensity for a peak ($A_{\mu}$)</th>
<th>Type of Gain</th>
<th>Analytical Result</th>
<th>Numerical Result</th>
<th>Percentage Difference ($\epsilon$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1 = \gamma_2 = 0.3$, $C_1 = 0.2$, $C_2 = 0.5$, $\delta_0 = 0.3$</td>
<td>0.8409</td>
<td>$GA_1$</td>
<td>1.745</td>
<td>1.737</td>
<td>0.46</td>
</tr>
<tr>
<td>$\gamma_1 = \gamma_2 = 0.3$, $C_1 = C_2 = 0.5$, $\delta_0 = 0.3$</td>
<td>0.3856</td>
<td>$GA_2$</td>
<td>2.007</td>
<td>2.007</td>
<td>1.13</td>
</tr>
<tr>
<td>$\gamma_1 = \gamma_2 = 0.01$, $C_1 = C_2 = 0.5$, $\delta_0 = 0.3$</td>
<td>0.3072</td>
<td>$GA_1$</td>
<td>4.756</td>
<td>4.756</td>
<td>0.23</td>
</tr>
<tr>
<td>$\gamma_1 = \gamma_2 = 0.3$, $C_1 = 0.2$, $C_2 = 1.0$, $\delta_0 = 0.3$</td>
<td>1.4779</td>
<td>$GA_2$</td>
<td>2.658</td>
<td>2.588</td>
<td>0.26</td>
</tr>
<tr>
<td>$\gamma_1 = \gamma_2 = 0.3$, $C_1 = 0.2$, $C_2 = 0.5$, $\delta_0 = 0.3$</td>
<td>0.6308</td>
<td>$GB_1$</td>
<td>1.375</td>
<td>1.351</td>
<td>1.75</td>
</tr>
<tr>
<td>$\gamma_1 = \gamma_2 = 0.3$, $C_1 = 0.2$, $C_2 = 0.5$, $\delta_0 = 0.3$</td>
<td>0.2866</td>
<td>$GB_2$</td>
<td>2.408</td>
<td>2.366</td>
<td>2.56</td>
</tr>
</tbody>
</table>

Table 1
Fig. 1. Histograms (30 bins) for probability density $P(x_n, t)$ for displacements $x_1(t)$ (blue) and $x_2(t)$ (orange) of the two oscillators for (i) $\Lambda = 0.25$, (ii) $\Lambda = 0.5$ and (iii) $\Lambda = 1.0$ at times (a) $t = 3.0$, (b) $t = 8.0$, (c) $t = 11.0$ and (d) $t = 14.3$. The parameters characterizing the oscillators and the ADN are $m_1 = 1.0, m_2 = 0.25, \gamma_1 = \gamma_2 = 0.3, k_1 = k_2 = 1.0, \zeta_1 = 0.2, \zeta_2 = 0.5, a_1 = d_1 = 1.0, a_2 = d_2 = 0.2, b_1 = e_1 = 0.2, b_2 = e_2 = 0.05, \alpha = 1.6, \beta = 0.6, \nu = 0.03, \delta = 0.3, \Omega = 0.5$.

with increase in $t$. Also, $\|x_1(t)\|_{\text{max}}$ and $\|x_2(t)\|_{\text{max}}$ are observed for slightly different $t$ values because of difference in the values of their phase angles. For example, in the case of $\Lambda = 0.5, \|x_1(t)\|_{\text{max}} = 1.48, 1.53, 1.58, \text{and } 1.60$ at $t = 8.0, 14.3, 20.5, \text{and } 26.9$, respectively; while the corresponding $\|x_2(t)\|_{\text{max}} = 1.80, 1.86, 1.91 \text{and } 1.93$ are found to occur at $t$ values nearly less by 0.1 from these. Second, an increase in noise intensity leads to an enhancement in the magnitude of the maximum displacement and a shift in the corresponding time to a higher value. For example, close to the completion of the first time-period, for $\Lambda = 0.0, 0.25, 0.5, \text{and } 1.0, \|x_1(t)\|_{\text{max}} = 1.40, 1.44, 1.53, \text{and } 1.60$ at $t = 13.5, 13.9, 14.3, \text{and } 15.0$, respectively. The corresponding values of $\|x_2(t)\|_{\text{max}}$ are $1.54, 1.64, 1.84, \text{and } 2.17$, respectively, at $t$ values less by 0.1 or so as compared to those listed for $\|x_1(t)\|_{\text{max}}$.

5. Numerical results and discussion

In order to carry out a thorough investigation of the effect of various parameters pertaining to the two oscillators and the noise on $C_n$ and $C_B_n$, we have first found $\Lambda_{cr}$ graphically by determining a reasonably precise value of $\Lambda_{cr}$ at which the $D(s = 0)$ plot crosses the $\Lambda_{cr}$ axis, for the specific set of the parameters. This has been then used as an upper limit for the range of $\Lambda_{cr}$ values, with compliance with (27), to obtain plots of the gains $G_A_n$ and $G_B_n, (n = 1, 2)$, for that case.

It may be mentioned that coupling between the two oscillators makes expression for $D(s)$, Eq. (B.2) together with Eqs. (A.1), to contain not only the parameters characterizing the ADN but also those belonging to both the oscillators at
Fig. 2. Displacements $x_1(t)$ (black dots) and $x_2(t)$ (red asterisks) as function of time for different $t$ values for (a) $\Lambda_{\mu} = 0.0$, (b) $\Lambda_{\mu} = 0.25$, (c) $\Lambda_{\mu} = 0.50$, (d) $\Lambda_{\mu} = 1.0$, for the oscillator and noise parameters listed in the caption of Fig. 1, except (a), where all the noise parameters are zero.

the same footing. Not only this, the numerators of $A_n$ and $B_n$, Eq. (24), for one oscillator involve parameters of the other oscillator as well. Consequently, the gains for an oscillator reasonably strongly depend on the parameters of both the oscillators as well as of the noise. However, rather than investigating the effect of all the parameters on the variation of $G_{An}$ and $G_{Bn}$ as function of the noise intensity $\Lambda_{\mu}$, we mainly concentrate on the influence of changes in the mass parameter, the coupling coefficients and the special case of zero value of the potential parameters. In order to show the effect of the non-modulated and modulated external periodic forces in proper perspective, $\Lambda_{\mu \xi}$ has been taken to be 1.0 in all the numerical calculations.

5.1. Effect of variation in $m$

The influence of variation in mass parameter of an oscillator on the gains $G_{An}$ and $G_{Bn}$, $(n = 1, 2)$, has been brought out by plotting the dependence of the gains on noise intensity for $m_1 = 1.0$ and $m_2 = 1.0, 0.5,$ and 0.25 in Fig. 3, keeping other typically selected parameters the same, which have been mentioned there. A look at the three figures here reveals that the presence of noise in the applied periodic signal leads to smaller gain, $G_{Bn}$, than that obtained in the same oscillator subjected to noise-free periodic force, $G_{An}$, which is understandable because the average influence of noise-modulation of the force will reduce its effectiveness. Nonetheless, the SR peaks are observed in all the cases presented here except $G_{A2}$ in Fig. 3(c) and the relative positions of all the curves are essentially the same.

The two oscillators considered in Fig. 3(a) differ from each other in the values of $C_1$ and $C_2$, and if these are interchanged then the curves $G_{A1}, G_{B1},$ and $G_{A2}, G_{B2}$ also get interchanged. Thus, the oscillator experiencing stronger coupling, has larger value of the gain when their masses are the same. A comparison of Fig. 3(a)–(c) shows that decrease in $m_2$ value shifts the position of the maximum in a gain to higher value of $\Lambda_{\mu}$. However, the magnitudes of the maximum value of the gains exhibit different behaviour: $G_{A1}$, $G_{B1}$, and $G_{A2}$ increase with decrease in $m_2$, while this is reduced in the case of $G_{B2}$. This observation can be explained by the fact that reduction in the $m_2$ value decreases the intrinsic energy, which, in turn, makes more energy available to enhance the output amplitude of oscillator 2 and hence its gain $G_{A2}$ and this effect is communicated to oscillator 1 through coupling. On the other hand, the averaging effect in the noise-modulated force contribution in the oscillator itself becomes stronger leading to decrease in $G_{B2}$.

Now onwards, we shall consider the oscillators with $m_1 = 1.0$ and $m_2 = 0.25$. Also, it may be pointed out that for $\Omega = 0.1, 0.01$, and 0.001 (the values to be used in Section 5.3), the gains exhibit a monotonically increasing trend for $\Lambda_{\mu} < (\Lambda_{\mu})_{cr} = 1.81$ and the peaks occur only for $\Lambda_{\mu} > (\Lambda_{\mu})_{cr}$. For example, in the case $\Omega = 0.01$ this happens around $\Lambda_{\mu} = 1.86$.

5.2. Effect of variation in coupling parameter

With a view to examine the effect of change in coupling strength on the gains, we have studied $G_{An}$ and $G_{Bn}$ as function of $\Lambda_{\mu}$ for different pairs of values of $C_1$ and $C_2$ rather than a continuous variation of either to make sure that
the inequality (27) is satisfied in each case. It has been found that the curves monotonically decrease without any peaks when both $C_1, C_2 > 0.7$, for which $(A_{\mu})_{cr} < 0.3$. Some typical cases for which SR is observed, are displayed in Fig. 4 and the parameter values used are included in the caption. From Fig. 4(a), we note that for $C_1 = C_2 = 0.2$, $(A_{\mu})_{cr} = 2.31$, the gains for oscillator 1, viz. $G_A_1$ and $G_B_1$, show peaks, while the gains $G_A_2$ and $G_B_2$ for oscillator 2 monotonically increase with $A_{\mu}$. When $C_1 = C_2 = 0.5$, for which $(A_{\mu})_{cr} = 0.78$, Fig. 4(b), all the four gains show quite well defined peaks: those corresponding to the case of noise-modulated periodic force at lower but the same value of $A_{\mu}$ and those pertaining to non-modulated periodic force at higher and the same value of $A_{\mu}$. Moreover, the gains for oscillator 1 have higher magnitude than those for oscillator 2. Also, from Fig. 4(c), for which $C_1 = 0.5$, $C_2 = 0.2$, and $(A_{\mu})_{cr} = 1.81$, we note that the plots for $G_A_2$ as well as $G_B_2$ monotonically increase with $A_{\mu}$, while the peaks for $G_A_1$ and $G_B_1$ occur for the $A_{\mu}$ values reasonably larger than the respective values for $C_1 = 0.2$, $C_2 = 0.5$ $(A_{\mu})_{cr} = 1.81$ projected in Fig. 3(c); these also have higher peak magnitude.

It may be mentioned that if both the $\gamma_1$ and $\gamma_2$ values are decreased from 0.3 to 0.01, then the $(A_{\mu})_{cr}$ values get somewhat reduced and the gains vs $A_{\mu}$ graphs show significant differences. Comparing Figs. 5(a) and 5(b) with Figs. 4(a) and 4(b), respectively, we note that all the gains for $\gamma_1 = \gamma_2 = 0.01$ show SR peaks, which have higher peak magnitudes and are also relatively sharp because the attenuation due to the external damping is reduced. The curves for $C_1 = 0.5$, $C_2 = 0.2$ in Figs. 5(c) and Fig. 4(c) present an interesting contrast. For smaller value of $\gamma_1$ and $\gamma_2$, $G_A_1$ monotonically
Fig. 4. Plots of $G_A$, $G_B$, $G_A^2$, and $G_B^2$ versus $\Lambda_\mu$ for (a) $C_1 = C_2 = 0.2$, (b) $C_1 = C_2 = 0.5$, and (c) $C_1 = 0.5$, $C_2 = 0.2$, and other parameters $\gamma_1 = \gamma_2 = 0.3$, $k_1 = k_2 = 1.0$, $a_1 = d_1 = 1.0$, $a_2 = d_2 = 0.2$, $m_1 = 1.0$, $m_2 = 0.25$, $b_1 = e_1 = 0.2$, $b_2 = e_2 = 0.05$, $\Omega = 0.5$, $\alpha = 1.6$, $\beta = 0.6$, $\nu = 0.3$, $\delta = 0.3$, $\Lambda_\mu = 1.0$.

5.3. Gains for the case $k_1 = k_2 = 0.0$

As pointed out in the paragraph after Eq. (26), the presence of the nonlinear term in the noise can make rectilinear motion of a particle to be oscillatory; we have studied this aspect in detail by considering the situation $k_1 = k_2 = 0.0$. The results for a typical case corresponding to different $\Omega$ values are depicted in Fig. 6. For the parameter values used here, the SR peaks are not observed if $\Omega = 0.5$, the value used in Figs. 3–5. Nonetheless, SR does occur and that for the lower values of the frequency of the applied periodic force because its intrinsic frequency of the particle having its origin in the presence of nonlinear term of the noise, like $(a_2 A_\mu y_1(t))$, will be less than the case when $k_1 \neq 0$. A perusal of Fig. 6(a)–(c) reveals that the values of $A_\mu$, for which, the four peaks appear are little different from each other for $\Omega = 0.1$ but the same when $\Omega = 0.01$ and $\Omega = 0.001$; these values increase as $\Omega$ is decreased. Besides, the values of all the gains get
enhanced by a factor of about 5 when $\Omega$ is lowered by a factor of 10. It may be mentioned that the graphs for the gains corresponding to non-modulated and modulated external forces for an oscillator appear to be closer with decrease in the $\Omega$ values and, hence, the $A_\mu$ scale for Fig. 6(c) has been taken to be quite different from the other two to make the separation between the plots to be noticeable.

It may be added that if we take $a_1 = d_1 = 1.0$, as has been done in Figs. 3–5, rather than 0.0 (Fig. 6), then the value of $(A_\mu)_{ct}$ turns out to be as small as 0.0126 instead of 0.951 and the plots for gains monotonically increase for $\Omega = 0.1$, increase for $\Omega = 0.001$ with the maximum magnitude at about $A_\mu = 0.0126$, and show peaks at $A_\mu = 0.0115$ (oscillator 1) and 0.0120 (oscillator 2) for $\Omega = 0.01$.

Furthermore, it is found that replacing $C_1$ and $C_2$ values from 0.2 and 0.5, respectively, by zero for both (implying absence of coupling) reduces $(A_\mu)_{ct}$ to 0.723 from 0.951 (Fig. 6) and the gains monotonically decrease from a high value at $A_\mu = 0$ and, thus, do not show any SR peaks for the $\Omega$ values used here. This brings out the fact that coupling plays an important role in inducing SR in the particles having rectilinear motion.

Next, if the values of both $\gamma_1$ and $\gamma_2$ are reduced from 0.3 to 0.01, keeping other parameters the same as in Fig. 6, then $(A_\mu)_{ct} = 1.0160$ and the plots for $\Omega = 0.1$ and $\Omega = 0.01$ show no peak for $A_\mu < (A_\mu)_{ct}$ while the graphs pertaining to $\Omega = 0.001$ have the SR peaks at $A_\mu = 1.0158$, with values that are nearly 19 times the magnitude of $G_{A_n}$ in Fig. 6(c) and approximately 16 times the $G_{B_n}$ there. In this case when $C_1 = C_2 = 0.0$, only monotonically decreasing curves with quite

Fig. 5. Same as Fig. 4 except $\gamma_1 = \gamma_2 = 0.01$, with (a) $C_1 = C_2 = 0.2$, (b) $C_1 = C_2 = 0.5$, and (c) $C_1 = 0.5, C_2 = 0.2$. 
large values of gains at $\Lambda = 0$, are obtained for $\Omega = 0.01$ and $\Omega = 0.001$, though the plots corresponding to $\Omega = 0.1$ have peaks for $\Lambda < 0.015 \ll (\Lambda_{\text{cr}})_{\text{cr}} = 0.866$.

6. Concluding remarks

Stochastic resonance in two coupled oscillators governed by fractional-order intrinsic and external damping under the influence of a multiplicative quadratic asymmetric dichotomous noise affecting the two potential parameters as well as the linear coupling coefficients and driven by a noise-modulated or non-modulated periodic force, has been studied. Exact expressions for the average output amplitude gains have been obtained in the long-time limit employing the commonly used Laplace transform method. Plots for these as function of noise intensity value lower than the relevant critical noise intensity satisfying stability condition, have been compared for typical values of various parameters to analyse essentially the effect of variation in mass as well as coupling parameters and the special case where both the potential parameters are taken to be zero, which, in turn, make the motion to be rectilinear. The main conclusions drawn are listed in the sequel.

1. For the chosen parameters, when $m_2$ values are reduced, the $\Lambda$ values at which the gains show peak become larger, and magnitudes of $G_A$, $G_B$ and $G_A$ increase while that of $G_B$ decreases. It means that the decrease in the intrinsic
energy of oscillator 2 due to reduction in its mass makes more energy available to increase its output amplitude and hence the gain and this effect is shared by the other oscillator because of coupling. Besides, the averaging effect in the displacement associated with the noise-modulated force in the oscillator itself becomes more significant.

2. When \(m_1 > m_2\), the plots for gains show well defined SR peaks only if both the coupling coefficients are not too large. For relatively very small values of \(\gamma_1\) and \(\gamma_2\), the gains exhibit sharp SR peaks with higher magnitudes because of lesser attenuation and, also, low dip valleys are observed for \(C_2 = 0.2\) and \(C_1 \geq 0.5\).

3. It is argued that the presence of the second-order term in the multiplicative noise associated with \(x(t)\) can make the rectilinear motion of a particle to be oscillatory so that the coupled particles with \(k_1 = k_2 = 0\) will behave as oscillators. It has been demonstrated that such a system shows SR at lower values of the frequency \(\Omega\) of the external periodic force and that coupling is essential for getting SR in this case.

Besides the focused work on stochastic resonance as summarized above, the numerical solutions for Eqs. (1) and (2) have been used to verify the reliability of the results found from the analytical expressions by considering some typical cases. Furthermore, these solutions have been used to study the time-evolution of the two oscillators much before attaining the steady state, by determining the probability density and displacement of the two oscillators in different situations.

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**CRediT authorship contribution statement**

**Vishwamittar:** Conceptualization, Formal analysis, Investigation, Methodology, Project administration, Resources, Software, Validation, Visualization, Writing - original draft, Writing - review & editing. **Priyanka Batra:** Resources, Software, Supervision, Writing - review & editing. **Ribhu Chopra:** Software, Supervision, Writing - review & editing.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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### Appendix A. Expressions for \(T_{jn}\)

The coefficients of \(Y_j(s)(j = 1 - 4)\), introduced in the system of linear equations (16)–(19) are given by the following expressions.

\[
\begin{align*}
T_{11} & = m_1s^\alpha + \gamma_1s^\beta + k_1 + a_2A_\mu, \\
T_{12} & = -(C_1 + b_2A_\mu), \\
T_{13} & = a_1 + a_2\delta_\mu, \\
T_{14} & = -(b_1 + b_2\delta_\mu), \\
T_{21} & = \Lambda_\mu T_{13}, \\
T_{22} & = \Lambda_\mu T_{14}, \\
T_{23} & = m_1(s + v_\mu)s^\alpha + \gamma_1(s + v_\mu)s^\beta + k_1 + a_1\delta_\mu + a_2(A_\mu + \delta_\mu^2), \\
T_{24} & = -(C_1 + b_1\delta_\mu + b_2(A_\mu + \delta_\mu^2)), \\
T_{31} & = -(C_2 + e_2A_\mu), \\
T_{32} & = m_2s^\alpha + \gamma_2s^\beta + k_2 + d_2A_\mu, \\
T_{33} & = -(e_1 + e_2\delta_\mu), \\
T_{34} & = d_1 + d_2\delta_\mu, \\
T_{41} & = \Lambda_\mu T_{33}, \\
T_{42} & = \Lambda_\mu T_{34}, \\
T_{43} & = -(C_2 + e_1\delta_\mu + e_2(A_\mu + \delta_\mu^2)).
\end{align*}
\]
and
\[ T_{44} = m^2_2 (s + v_\mu)^2 + \gamma_2 (s + v_\mu)^2 + k_2 + d_1 \delta_\mu + d_2 (A_\mu + \delta_\mu^2). \]

**Appendix B. The Coefficients Appearing in Eqs. (20) and (21)**

The coefficients \( H_4a(s) \) and \( H_{26}(s) \), used in writing the expressions for \( Y_1(s) \) and \( Y_2(s) \), have the following meaning:

\[
H_{4a}(s) = \frac{1}{D(s)} \left[ T_{42}(T_{13}T_{24} - T_{14}T_{23} + T_{23}T_{34} - T_{24}T_{33} + T_{43}(-T_{12}T_{24} + T_{14}T_{22} - T_{22}T_{34} + T_{24}T_{32}) + T_{44}(-T_{12}T_{23} - T_{13}T_{22} + T_{22}T_{33} - T_{23}T_{32}) \right].
\]

(B.1)

\[
H_{18}(s) = \frac{1}{D(s)} \left[ T_{12}(-T_{23}T_{34} + T_{24}T_{33} - T_{33}T_{44} + T_{34}T_{43}) + T_{13}(T_{22}T_{34} - T_{24}T_{32} + T_{32}T_{44} - T_{34}T_{42}) + T_{14}(-T_{22}T_{33} + T_{23}T_{32} - T_{32}T_{43} + T_{33}T_{42}) \right].
\]

(B.2)

\[
H_{2a}(s) = \frac{1}{D(s)} \left[ T_{41}(-T_{13}T_{24} + T_{14}T_{23} - T_{23}T_{34} + T_{24}T_{33}) + T_{43}(T_{11}T_{24} - T_{14}T_{21} + T_{21}T_{34} - T_{24}T_{31}) + T_{44}(-T_{11}T_{23} + T_{13}T_{21} - T_{21}T_{33} + T_{23}T_{31}) \right].
\]

(B.3)

and

\[
H_{26}(s) = \frac{1}{D(s)} \left[ T_{11}(T_{23}T_{34} - T_{24}T_{33} + T_{33}T_{44} - T_{34}T_{43}) + T_{13}(-T_{21}T_{34} + T_{24}T_{31} - T_{31}T_{44} + T_{34}T_{41}) + T_{14}(T_{21}T_{33} - T_{23}T_{31} + T_{31}T_{43} - T_{33}T_{41}) \right].
\]

(B.4)

Here,

\[
D(s) = (T_{11}T_{22} - T_{12}T_{21})(T_{33}T_{44} - T_{34}T_{43}) + (T_{11}T_{24} - T_{14}T_{21})(T_{32}T_{43} - T_{33}T_{42}) + (T_{12}T_{23} - T_{13}T_{22})(T_{31}T_{44} - T_{34}T_{41}) + (T_{13}T_{21} - T_{11}T_{23})(T_{32}T_{44} - T_{34}T_{42}) + (T_{14}T_{23} - T_{13}T_{24})(T_{31}T_{42} - T_{32}T_{41}) + (T_{12}T_{24} - T_{14}T_{22})(T_{31}T_{43} - T_{33}T_{41}).
\]

(B.5)

**References**


